

SUPERCRITICAL NONLINEAR WAVE EQUATIONS: QUASI-PERIODIC SOLUTIONS AND ALMOST GLOBAL EXISTENCE

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ABSTRACT. We construct time quasi-periodic solutions and prove almost global existence for the energy supercritical nonlinear wave equations on the torus in arbitrary dimensions. This is an application of the geometric L^2 methods introduced in [W1, 2].

1. Introduction and statement of the Theorems

We consider *real* valued solutions to the nonlinear wave equation on the d -torus $\mathbb{T}^d = [0, 2\pi)^d$:

$$\frac{\partial^2 v}{\partial t^2} - \Delta v + v + v^{p+1} + H(x, v) = 0 \quad (p \geq 1, p \in \mathbb{N}), \quad (1.1)$$

with periodic boundary conditions: $v(t, x) = v(t, x + 2j\pi)$, $x \in [0, 2\pi)^d$ for all $j \in \mathbb{Z}^d$ and $v \in \mathbb{R}$; $H(x, v)$ is analytic in (x, v) and has the expansion:

$$H(x, v) = \sum_{m=p+2}^{\infty} \alpha_m(x) v^m,$$

where α_m as a function on \mathbb{R}^d is $(2\pi)^d$ periodic and real and analytic in a strip of width $\mathcal{O}(1)$ for all m . The integer p in (1.1) is *arbitrary*.

We use the standard ODE technique to write (1.1) as a first order equation in t . Let

$$D = \sqrt{-\Delta + 1} \quad (1.2)$$

and

$$u = (v, -D^{-1} \frac{\partial v}{\partial t}) \in \mathbb{R}^2. \quad (1.3)$$

Identifying \mathbb{R}^2 with \mathbb{C} , we then obtain the corresponding first order equation

$$i \frac{\partial u}{\partial t} = Du + D^{-1} \left[\left(\frac{u + \bar{u}}{2} \right)^{p+1} + H(x, \frac{u + \bar{u}}{2}) \right]. \quad (1.4)$$

Using Fourier series, the solutions to the linear equation:

$$i \frac{\partial u}{\partial t} = Du$$

are linear combinations of eigenfunction solutions of the form:

$$e^{-i(\sqrt{j^2+1})t} e^{ij \cdot x}, \quad j \in \mathbb{Z}^d,$$

where $j^2 = |j|^2$ and \cdot is the usual inner product. These solutions are either periodic or quasi-periodic in time.

Applying the techniques developed in [W1, 2], we construct quasi-periodic solutions and prove almost global existence for a class of smooth solutions to Cauchy problems for the nonlinear wave equation in (1.4, 1.1). As in [W1, 2], the method hinges on analyzing the geometry of the nonlinearity relative to the characteristics and does not make use of conservation laws. Therefore the results hold both for the focusing and the defocusing cases and have particular relevance when global solutions are not known a priori by using energy conservation.

The nonlinear Fourier series

To proceed, let $u^{(0)}$ be a solution of finite number of frequencies, b frequencies, to the linear equation:

$$i \frac{\partial u^{(0)}}{\partial t} = Du^{(0)}, \tag{1.5}$$

$$u^{(0)}(t, x) = \sum_{k=1}^b a_k e^{-i(\sqrt{j_k^2+1})t} e^{ij_k \cdot x}.$$

For the nonlinear construction, it is useful to add a dimension for each frequency in time and view $u^{(0)}$ as a function on $\mathbb{T}^b \times \mathbb{T}^d = \mathbb{T}^{b+d} \supset \mathbb{T}^d$. Henceforth $u^{(0)}$ adopts the form:

$$\begin{aligned} u^{(0)}(t, x) &= \sum_{k=1}^b a_k e^{-i(\sqrt{j_k^2+1})t} e^{ij_k \cdot x} \\ &:= \sum_{k=1}^b \hat{u}(-e_k, j_k) e^{-i(e_k \cdot \omega^{(0)})t} e^{ij_k \cdot x}, \end{aligned}$$

where $e_k = (0, 0, \dots, 1, \dots, 0) \in \mathbb{Z}^b$ is a unit vector, with the only non-zero component in the k th direction, $\omega^{(0)} = \{\sqrt{j_k^2+1}\}_{k=1}^b$ ($j_k \neq 0$) and $\hat{u}(-e_k, j_k) = a_k$. Therefore $u^{(0)}$ has Fourier support

$$\text{supp } \hat{u}^{(0)} = \{(-e_k, j_k), k = 1, \dots, b\} \subset \mathbb{Z}^{b+d}, \tag{1.6}$$

where $j_k \neq j_{k'}$ if $k \neq k'$.

For the nonlinear equation (1.1), we seek quasi-periodic solutions with b frequencies in the form of a *nonlinear* space-time Fourier series:

$$u(t, x) = \sum_{(n, j)} \hat{u}(n, j) e^{in \cdot \omega t} e^{ij \cdot x}, \quad (n, j) \in \mathbb{Z}^{b+d}, \quad (1.7)$$

with the frequency $\omega \in \mathbb{R}^b$ to be determined. This is the well-known frequency-amplitude modulation fundamental to nonlinear equations. We note that the corresponding linear solution $u^{(0)}$ has *fixed* frequency $\omega = \omega^{(0)} = \{\sqrt{j_k^2 + 1}\}_{k=1}^b \in \mathbb{R}^b$, which are eigenvalues of the operator D in (1.2).

In the Fourier space \mathbb{Z}^{b+d} , the support of the solution in the form (1.7) to the linear equation (1.5) and its complex conjugate are by definition, the bi-characteristics \mathcal{C} , $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ with

$$\mathcal{C}_{\pm} = \{(n, j) \in \mathbb{Z}^{b+d} \mid \pm n \cdot \omega^{(0)} + \sqrt{j^2 + 1} = 0\}. \quad (1.8)$$

\mathcal{C} is the support of the solution to the linear equation (1.5) in the form (1.7) and is the resonant or singular set for the nonlinear equation (1.1). We consider \mathcal{C} as the restriction to \mathbb{Z}^{b+d} of the corresponding manifold, the hyperboloids on \mathbb{R}^{b+d} . So \mathcal{C} is in general a manifold of singularities and not just isolated points.

We say that a solution to the linear equation (1.5) is *generic* if its spatial Fourier support, which is a subset of $(\mathbb{Z}^d)^b$, satisfies the genericity conditions (i-iii) in sect. 2. In that case we also say that its real part, which is a real solution to the linear wave equation:

$$\frac{\partial^2 v^{(0)}}{\partial t^2} - \Delta v^{(0)} + v^{(0)} = 0, \quad (1.9)$$

is *generic*. As in [W1, 2], the main role of the generic linear solutions is to ensure good geometry in order to create a spectral gap. We note here that the non-generic set to be detailed in sect. 2 is of codimension 1 in $(\mathbb{R}^d)^b$.

The two main results are the following:

Theorem 1. *Assume*

$$v^{(0)}(t, x) = \operatorname{Re} \sum_{k=1}^b a_k e^{-i(\sqrt{j_k^2 + 1})t} e^{ij_k \cdot x}$$

is generic and $a = \{a_k\} \in (0, \delta]^b = \mathcal{B}(0, \delta)$ and p is even. Then there exist $C, c > 0$, such that for all $\epsilon \in (0, 1)$, there exists $\delta_0 > 0$ and for all $\delta \in (0, \delta_0)$ a Cantor set \mathcal{G} with

$$\operatorname{meas} \{\mathcal{G} \cap \mathcal{B}(0, \delta)\} / \delta^b \geq 1 - C\epsilon^c. \quad (1.10)$$

For all $a \in \mathcal{G}$, there is a quasi-periodic solution of b frequencies to the nonlinear wave equation (1.1):

$$v(t, x) = \operatorname{Re} \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + o(\delta^{3/2}), \quad (1.11)$$

with basic frequencies $\omega = \{\omega_k\}$ satisfying

$$\omega_k = \sqrt{j_k^2 + 1} + \mathcal{O}(\delta^p).$$

The remainder $o(\delta^{3/2})$ is in an analytic norm about a strip of width $\mathcal{O}(1)$ on \mathbb{T}^{b+d} .

For the Cauchy problem, we set $H = 0$ in (1.1) as is the custom in PDE. It is convenient to add a parameter and consider initial data of size one. So we have the following Cauchy problem on \mathbb{T}^d :

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \Delta v + v + \delta v^{p+1} = 0, \\ v(t=0) = v_0, v'(t=0) = \tilde{v}_0, \end{cases}$$

where δ is the parameter.

Theorem 2. *Let $v_0 = v_1 + v_2$ and $\tilde{v}_0 = \tilde{v}_1 + \tilde{v}_2$ be real valued. Assume $(v_1, -D^{-1}\tilde{v}_1) := u_1$ is generic and $\|(v_2, -D^{-1}\tilde{v}_2)\| := \|u_2\| = \mathcal{O}(\delta)$, where $\|\cdot\|$ is an analytic norm about a strip of width $\mathcal{O}(1)$ on \mathbb{T}^d . Let $\mathcal{B}(0, 1) = (0, 1]^b$, where b is the dimension of the Fourier support of u_1 . Let p be even and set $H = 0$ in (1.1). Then for all $A > 1$, there exist an open set $\mathcal{A} \subset \mathcal{B}(0, 1)$ of positive measure and $\delta_0 > 0$, such that for all $\delta \in (-\delta_0, \delta_0)$, if $\{|\hat{u}_1|\} \in \mathcal{A}$, then (1.1) has a unique solution $v(t)$ for $|t| \leq \delta^{-A}$ satisfying $v(t=0) = v_0$, $v'(t=0) = \tilde{v}_0$ and $\|v(t)\| + \|v'(t)\| \leq \|v_0\| + \|\tilde{v}_0\| + \mathcal{O}(\delta)$. Moreover, if $v_2 = \tilde{v}_2 = 0$, then $\operatorname{meas} \mathcal{A} \rightarrow 1$ as $\delta \rightarrow 0$.*

Remark. The geometric concept of generic linear solutions remains valid for odd p , cf. Proposition 2 in sect. 2. It is determined entirely by the leading order nonlinear term u^{p+1} . Once there is good geometry, the assumption of even p is sufficient but *not* necessary to ensure a spectral gap leading to amplitude-frequency modulations, cf. proof of Proposition 1. We note also that the parity in v of the higher order terms do not matter as they are treated as perturbations.

Theorem 1 is the first general existence results on quasi-periodic solutions to the nonlinear wave equation (1.1) in arbitrary dimensions. Previously quasi-periodic solutions were only constructed in one dimension, mostly using as parameter the positive mass m in the linear wave equation:

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + mv = 0, \quad (1.12)$$

which gives an eigenvalue set $\{\sqrt{j^2 + m}, j \in \mathbb{Z}\}$ close to the set of integers, see [B3, GR, P] and cf. also [K, Way] in a related context. For almost all m , this set is linearly independent over the integers. This property does not have higher dimensional analogues and was a fundamental obstacle.

The change of paradigm in [W1, 2] is *not* to prevent the eigenvalues to become linearly dependent over the integers and instead study the geometry of the nonlinearity relative to this set of zeroes – the bi-characteristics. In equation (1.1), the mass m is *fixed* at 1 and we use L^2 methods to create a spectral gap from the good geometry entailed by generic linear solutions as in [W1, 2].

In dimensions $d \geq 3$, when $H = 0$ and $p > \frac{4}{d-2}$, equation (1.1) is energy supercritical and global existence is generally unknown. Theorem 2 provides a first result on almost global existence and uniqueness of solutions for a class of smooth solutions to Cauchy problems. (For results with the mass m as a parameter and existence time slightly longer than the local one, namely $\delta^{-(1+2/d)}$ up to a $|\log \delta|^{-1}$ factor, see [D1]. For results in one dimension with the mass m as a parameter, see [BaGr, B1]; and with a quasi-linear term, see [D2].)

It is important to remark that it is the *very same* spectral gap that gives both existence of quasi-periodic solutions and almost global existence to Cauchy problems. This is made possible by the L^2 methods.

2. The generic linear solutions

Using the ansatz (1.7), (1.4) becomes

$$\text{diag } (n \cdot \omega + \sqrt{j^2 + 1})\hat{u} + \text{diag } (1/\sqrt{j^2 + 1})[(\frac{\hat{u} + \hat{\bar{u}}}{2})^{*(p+1)} + \sum_{m=2}^{\infty} \hat{\alpha}_m * (\frac{\hat{u} + \hat{\bar{u}}}{2})^{*(p+m)}] = 0, \quad (2.1)$$

where $(n, j) \in \mathbb{Z}^{b+d}$, $\omega \in \mathbb{R}^b$ is to be determined and

$$|\hat{\alpha}_m(\ell)| \leq C e^{-c|\ell|} \quad (C, c > 0)$$

for all m . From now on we work with (2.1), for simplicity we drop the hat and write u for \hat{u} and \bar{u} for $\hat{\bar{u}}$ etc.

We complete (2.1) by writing the equation for the complex conjugate. So we have

$$\begin{cases} \text{diag } (n \cdot \omega + \sqrt{j^2 + 1})u + \text{diag } (1/\sqrt{j^2 + 1})[(\frac{u + \bar{u}}{2})^{*(p+1)} + \sum_{m=2}^{\infty} \alpha_m * (\frac{u + \bar{u}}{2})^{*(p+m)}] = 0, \\ \text{diag } (-n \cdot \omega + \sqrt{j^2 + 1})\bar{u} + \text{diag } (1/\sqrt{j^2 + 1})[(\frac{u + \bar{u}}{2})^{*(p+1)} + \sum_{m=2}^{\infty} \alpha_m * (\frac{u + \bar{u}}{2})^{*(p+m)}] = 0. \end{cases} \quad (2.2)$$

We seek solutions close to the linear solution $u^{(0)}$ of b frequencies, $\text{supp } u^{(0)} = \{(-e_k, j_k), k = 1, \dots, b\}$, with frequencies $\omega^{(0)} = \{\sqrt{j_k^2 + 1}\}_{k=1}^b$ ($j_k \neq 0$) and small

amplitudes $a = \{a_k\}_{k=1}^b$ satisfying $\|a\| = \mathcal{O}(\delta) \ll 1$. Denote the left side of (2.2) by $F(u, \bar{u})$.

Linearizing at $(u^{(0)}, \bar{u}^{(0)})$, we are led to study the linearized operator $F'(u^{(0)}, \bar{u}^{(0)})$ on $\ell^2(\mathbb{Z}^{b+d}) \times \ell^2(\mathbb{Z}^{b+d})$ with

$$F' = D' + A + \mathcal{O}(\delta^{p+1}), \quad (2.3)$$

where

$$D' = \begin{pmatrix} \text{diag}(n \cdot \omega^{(0)} + \sqrt{j^2 + 1}) & 0 \\ 0 & \text{diag}(-n \cdot \omega^{(0)} + \sqrt{j^2 + 1}) \end{pmatrix} \quad (2.4)$$

and

$$\begin{aligned} A &= \left(\frac{p+1}{2^{p+1}}\right) \text{diag} \begin{pmatrix} 1/\sqrt{j^2 + 1} & 0 \\ 0 & 1/\sqrt{j^2 + 1} \end{pmatrix} \begin{pmatrix} (u^{(0)} + \bar{u}^{(0)})^{*p} & (u^{(0)} + \bar{u}^{(0)})^{*p} \\ (u^{(0)} + \bar{u}^{(0)})^{*p} & (u^{(0)} + \bar{u}^{(0)})^{*p} \end{pmatrix} \quad (p \geq 1), \\ &:= \left(\frac{p+1}{2^{p+1}}\right) \text{diag} \begin{pmatrix} D^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} A_0. \end{aligned} \quad (2.5)$$

The main purpose of this section is to prove the following *non-perturbative* spectral gap proposition:

Proposition 1. *Assume $u^{(0)}$ is generic satisfying the genericity conditions (i, ii) below. Let P_{\pm} be the projection on \mathbb{Z}^{b+d} onto \mathcal{C}^{\pm} defined in (1.8) and*

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}. \quad (2.6)$$

Let $a = \{a_k\}_{k=1}^b \in \mathcal{B}(0, 1) = (0, 1]^b$. There exist $C, c > 0$ such that for all $\epsilon > 0$, the operator A_0 in (2.5) satisfies

$$\|[PA_0P]^{-1}\| \leq [\epsilon \|a\|_{\infty}^p]^{-1}$$

on a set $\mathbb{B} \subset \mathcal{B}$ satisfying

$$\text{meas } \mathbb{B} / \|a\|_{\infty}^b > 1 - C\epsilon^c,$$

if p is even.

Remarks. 1. Proposition 1 only relies on the geometric properties of $u^{(0)}$ and $\|a\|_{\infty}$ is a mere scaling factor.

2. Using algebraic number properties of the set in (1.8) and the Schur complement reduction [S1, 2], this spectral gap will be converted into a corresponding gap for the

full linearized operator F' when restricting to $\|n\|_1 \leq N$ and N could depend on δ , cf. proofs of Theorems 1 and 2.

The generic linear solutions

To define generic $u^{(0)}$, we need to analyze the convolution matrix A_0 defined in (2.5). Let

$$\Gamma = \text{supp} [(u^{(0)} + \bar{u}^{(0)})^{*p}] = \{(\Delta n, \Delta j)\} \subset \mathbb{Z}^{b+d}, \quad (2.7)$$

with

$$\text{supp } u^{(0)} = \{(-e_k, j_k)\}_{k=1}^b, j_k \neq j_{k'} \text{ if } k \neq k'.$$

So

$$\begin{aligned} \Delta n &= -\sum (p_k e_k - p_{k'} e_{k'}), \\ \Delta j &= \sum (p_k j_k - p_{k'} j_{k'}), \\ p_k, p_{k'} &\geq 0, \sum (p_k + p_{k'}) = p, \\ &\text{where all sums are for } k, k' = 1, \dots, b. \end{aligned} \quad (2.8)$$

We note that when p is even, Γ contains the neutral element $I = (0, 0)$: $\Gamma \ni I$.

Let

$$B = \frac{(d+1)(d+2)}{2} + b(b+1). \quad (2.9)$$

The significance of B will be observed in Proposition 2 below. Here it suffices to note that $B < 3(d^2 + b^2)$. Let

$$\mathcal{A} = \bigcup \Gamma_1 \cdot \dots \cdot \Gamma_B := \bigcup \prod_{i=1}^B \Gamma_i, \quad (2.10)$$

where Γ_i is Γ or I , and the multiplication \cdot stands for multiplication of Fourier series and the union is over all choices of Γ_i .

Elements of \mathcal{A} are of the form:

$$\mathcal{A} \ni (\Delta n, \Delta j) = \sum_{i \leq B} (\Delta n^{(i)}, \Delta j^{(i)}), \quad (2.11)$$

where $(\Delta n^{(i)}, \Delta j^{(i)}) \in \Gamma$ or equals to I . So any finite product with at most B factors is in \mathcal{A} . For σ in \mathcal{A} , write $|\sigma|$ for its length.

For $(\Delta n, \Delta j) \in \mathcal{A}$, let $\Delta j^{(m)}$, $m = 1, \dots, d$, be the m^{th} component of $\Delta j \in \mathbb{Z}^d$ and identify the set of unordered pairs $\ell := (m, m')$, $m \neq m'$, $m, m' = 1, \dots, d$ with the set

$\{1, 2, \dots, d(d-1)/2\}$. We define the following functions:

$$\begin{aligned}
a_m &= 4[\Delta j^{(m)}]^2 - 4[\Delta n \cdot \omega^{(0)}]^2, & m &= 1, \dots, d, \\
b_\ell &:= b_{mm'} = 8\Delta j^{(m)} \Delta j^{(m')}, & m, m' &= 1, \dots, d, m \neq m', \\
c_m &= [4\Delta j^2 - 4(\Delta n \cdot \omega^{(0)})^2] \Delta j^{(m)}, & m &= 1, \dots, d, \\
L &= \{\{a_m\}, \{b_\ell\}, \{c_m\}\} \in \mathbb{Z}^{d(d+3)/2}, \\
W &= [\Delta j^2 - (\Delta n \cdot \omega^{(0)})^2]^2 - 4[\Delta n \cdot \omega^{(0)}]^2.
\end{aligned} \tag{2.12}$$

We also isolate a subset \mathcal{G} of \mathcal{A} :

$$\mathcal{A} \supset \mathcal{G} := \{\alpha(e_{k'} - e_k, j_k - j_{k'}), \alpha(-e_k, j_k); \alpha \in \mathbb{Z} \setminus \{0\}, k, k' = 1, \dots, b, k \neq k'\},$$

and let

$$\mathcal{S} = \text{supp } u^{(0)} \cup \text{supp } \bar{u}^{(0)} \subset \mathbb{Z}^{b+d}. \tag{2.13}$$

Definition. $u^{(0)}$ of b frequencies is *generic* if its Fourier support $\{(-e_k, j_k)\}_{k=1}^b \subset \mathbb{Z}^{b+d}$, where $j_k \neq j_{k'}$ if $k \neq k'$ satisfies:

(i) For all $(\Delta n, \Delta j) \in \mathcal{A} \setminus (0, 0)$,

$$\begin{aligned}
\Sigma_\pm &= \Delta j^2 \pm [\Delta n \cdot \omega^{(0)}]^2 \neq 0, \\
W &= [\Delta j^2 - (\Delta n \cdot \omega^{(0)})^2]^2 - 4[\Delta n \cdot \omega^{(0)}]^2 \neq 0,
\end{aligned}$$

where $\omega^{(0)} = \{\sqrt{j_k^2 + 1}\}_{k=1}^b$.

(ii) Let $B' = \frac{(d+1)(d+2)}{2}$. The L defined in (2.10) is therefore a $(B' - 1)$ -vector. For any $\sigma \subset \mathcal{A} \setminus \mathcal{G}$ with $|\sigma| = B'$ and $\Delta j \neq 0$ identically for any $(\Delta n, \Delta j) \in \sigma$, the $B' \times B'$ matrix $[[L, W]]_{B' \times B'}$,

$$[[L, W]]_{B' \times B'} := \begin{pmatrix} L_1^{(1)} & L_1^{(2)} & \cdots & L_1^{(B'-1)} & W_1 \\ L_2^{(1)} & L_2^{(2)} & \cdots & L_2^{(B'-1)} & W_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{B'}^{(1)} & L_{B'}^{(2)} & \cdots & L_{B'}^{(B'-1)} & W_{B'} \end{pmatrix},$$

where for each $i = 1$ to B' , L_i and W_i are as defined in (2.10), satisfies the following properties:

Let $[[L]]_{B' \times (B'-1)}$ be the $B' \times (B' - 1)$ submatrix of $[[L, W]]_{B' \times B'}$. For any $1 \leq \rho \leq B' - 1$ and any $(\rho + 1) \times \rho$ submatrix $[[L]]_{(\rho+1) \times \rho}$ of $[[L]]_{B' \times (B'-1)}$, if none of

the $\rho \times \rho$ submatrix of $[[L]]_{(\rho+1) \times \rho}$ has zero determinant, then the corresponding $(\rho+1) \times (\rho+1)$ matrix $[[L, W]]_{(\rho+1) \times (\rho+1)}$ satisfies

$$D = \det[[L, W]]_{(\rho+1) \times (\rho+1)} \neq 0,$$

where W is the corresponding $(\rho+1)$ vector.

(iii)

$$\text{supp } (u^{(0)} + \bar{u}^{(0)})^{*p+1} \cap \{\mathcal{C} \setminus \mathcal{S}\} = \emptyset.$$

Remark. The plus part of the first condition in (i) prevents pure translations in time. This is a recurrent condition, which is almost necessary. The second condition excludes, in particular, $j = 0$ to be a solution. (iii) is for the analysis in the Newton scheme.

Lemma. *Let f be the functions defined by the contrary of (iii). Then the non-generic set: $\Omega := \{\Sigma_{\pm} = 0\} \cup \{D = 0\} \cup \{f = 0\}$ has codimension 1 in $(\mathbb{R}^d)^b$.*

Proof. The non-generic set Ω is defined by a finite number of analytic functions. It follows readily from the preparation theorem that the set $\{\Sigma_{\pm} = 0\} \cup \{f = 0\}$ has codimension 1. Setting $\Delta j = 0$ in (2.12) gives $a_m = -[\Delta n \cdot \omega^{(0)}]^2$, $b_{\ell} = c_m = 0$ and $W = [\Delta n \cdot \omega^{(0)}]^4$. For $(\Delta n, \Delta j) \in \mathcal{A} \setminus \mathcal{G}$, $\Delta n \cdot \omega^{(0)}$ is not identically zero and not a constant function. So W cannot be linearly dependent on L . It follows that the D defined in (ii) are not identically zero. The preparation theorem then gives that $\{D = 0\}$ has codimension 1. \square

Size of connected sets on the bi-characteristics

Let (n, j) and (n', j') be two points in \mathbb{Z}^{b+d} , $(n, j) \neq (n', j')$. We say that (n, j) and (n', j') are *connected* if the difference $(n - n', j - j')$ is in the algebra generated by Γ , cf. (2.7, 2.11). We say that a set S in \mathbb{Z}^{b+d} is *connected* if for any $(n, j) \in S$, $\exists (n', j') \in S$, $(n', j') \neq (n, j)$, such that $(n - n', j - j')$ is in the algebra generated by Γ . The number of elements in S is its size.

Following is our main geometric proposition:

Proposition 2. *Assume $u^{(0)}$ is generic satisfying the genericity conditions (i, ii). The connected sets on the hyperboloids \mathcal{C} defined in (1.8) are of sizes at most $B = \frac{(d+1)(d+2)}{2} + b(b+1) < 3(d^2 + b^2)$ as in (2.9).*

Proof. Assume there is a connected set S on the hyperboloid \mathcal{C} , $S \subset \mathcal{C}$ of size $B + 1$. Privilege an (arbitrary) element in S and name it (n, j) . The point (n, j) satisfies one of the two equations:

$$\pm n \cdot \omega^{(0)} + \sqrt{j^2 + 1} = 0. \tag{2.14}$$

All other elements $(n', j') \in S \subset \mathcal{C}$, $(n', j') \neq (n, j)$, can be written as

$$(n', j') = (n + \Delta n, j + \Delta j)$$

for some $(\Delta n, \Delta j) \in \mathcal{A} \setminus (0, 0)$ and satisfies one of the two equations:

$$\pm(n + \Delta n) \cdot \omega^{(0)} + \sqrt{(j + \Delta j)^2 + 1} = 0. \quad (2.15)$$

Equations defined by (2.15) form a system of B equations.

For each equation defined by (2.15), subtracting or adding (2.14) to eliminate the n variables. Rearranging the terms and squaring lead to the quadratic equations in $j = (j_1, j_2, \dots, j_d) \in \mathbb{Z}^d \subset \mathbb{R}^d$ of d variables:

$$\sum_{m=1}^d a_m j_m^2 + \sum_{m \neq m'} b_{mm'} j_m j_{m'} + \sum_{m=1}^d c_m j_m + W = 0, \quad (2.16)$$

where $a_m, b_{mm'}, c_m$ and W are defined according to (2.12). (2.16) defines a system of B quadratic equations.

We differentiate two types of equations in (2.16). The first type originates from $(\Delta n, \Delta j) \in \mathcal{A} \setminus \mathcal{G}$. To control this type of equations, we consider the corresponding “linearized” system of equations in $d(d+3)/2 = B' - 1$ variables $\{x_m\}_{m=1}^d, \{y_\ell\}_{\ell=1}^{d(d-1)/2}$ and $\{z_m\}_{m=1}^d$:

$$\sum_{m=1}^d a_m x_m + \sum_{\ell=1}^{d(d-1)/2} b_\ell y_\ell + \sum_{m=1}^d c_m z_m + W = 0, \quad (2.17)$$

where we have identified the unordered pairs (m, m') with ℓ as in (2.12). In order for the corresponding quadratic system in (2.16) to have a solution, it is necessary that the system (2.17) have a solution. The genericity conditions (i, ii) give that the system defined by (2.17) has a solution for at most $B' - 1$ equations.

The rest of the equations in (2.16) forms the second type $(\Delta n, \Delta j) \in \mathcal{G}$ and corresponds to the “degenerate case”, as setting $\Delta j = 0$, $\Delta n \cdot \omega^{(0)}$ is a constant equal to 0 or α . Assume there exist $\alpha \neq \beta \neq \gamma \neq 0$ such that $\alpha(e_{k'} - e_k, j_k - j_{k'})$, $\beta(e_{k'} - e_k, j_k - j_{k'})$ and $\gamma(e_{k'} - e_k, j_k - j_{k'})$ appear in (2.16), then the quadratic in j terms as well as the $[\Delta n \cdot \omega^{(0)}]^2$ term of the equations can be eliminated after pairwise subtraction as the coefficients are homogeneous of order 2. Since $\alpha \neq \beta \neq \gamma \neq 0$ and $\Sigma_- \neq 0$ from (i), this leads to 2 incompatible linear equations. Similar arguments hold for $\alpha(-e_k, j_k)$, $\beta(-e_k, j_k)$ and $\gamma(-e_k, j_k)$ with $\alpha \neq \beta \neq \gamma \neq 0$. So there can be at most $b(b-1) + 2b = b(b+1)$ such equations.

Combining the above two types of reasoning we reach the conclusion. \square

Remark. We note that for the Schrödinger case in [W1, 2], the analogue of (2.16) is a system of linear equations and the sizes of connected sets are at most $d + 2$ in the complement of the tangential sites $\{j_1, j_2, \dots, j_b\} \subset \mathbb{Z}^d$, independent of the number of frequencies b . The system in (2.16), on the other hand, is quadratic with in general non-integer (square roots) coefficients.

Proof of Proposition 1. Using Proposition 2, the connected sets on the hyperboloids \mathcal{C} are of sizes at most $B < 3(d^2 + b^2)$. So the matrix PA_0P decomposes into bloc diagonal matrices of sizes at most $B \times B$. The matrix entries are functions of the amplitude $a = \{a_k\}_{k=1}^b$. To prove the proposition, it suffices that the determinants of these finite matrices are not constants. Therefore it suffices to restrict to the periodic case with $a_1 \neq 0$ and $a_k = 0$ for $k \neq 1$.

We observe that

$$PA_0P = \left[P \begin{pmatrix} (u^{(0)} + \bar{u}^{(0)})^{*2} & 0 \\ 0 & (u^{(0)} + \bar{u}^{(0)})^{*2} \end{pmatrix} P \right]^{p/2-1} P \begin{pmatrix} (u^{(0)} + \bar{u}^{(0)})^{*2} & (u^{(0)} + \bar{u}^{(0)})^{*2} \\ (u^{(0)} + \bar{u}^{(0)})^{*2} & (u^{(0)} + \bar{u}^{(0)})^{*2} \end{pmatrix} P,$$

where we used that P commutes with bloc diagonal matrices. In the periodic case, the above matrix further decomposes into products of direct sums of 3×3 bloc diagonal matrices of the forms

$$a_1^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, a_1^2 \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, a_1^2 \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } a_1^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

cf. proof of Proposition 2 regarding the degenerate case. So the determinant of PA_0P is not a constant polynomial in a , which gives the measure estimate and completes the proof. \square

Remark. We note that even p is only used to ensure that the diagonal is non-zero in the periodic case.

3. Proofs of Theorems 1 and 2

We convert the spectral gap in Proposition 1 into a spectral gap for the full linearized operator F' defined in (2.3-2.5) when restricting the time harmonics $n \in \mathbb{Z}^b$. So let F'_N be the matrix operator:

$$\begin{aligned} F'_N(n, j; n', j') &= F'(n, j; n', j'), & \text{if } \|n\|_1, \|n'\|_1 \leq N, \\ &= 0, & \text{otherwise.} \end{aligned} \tag{3.1}$$

We have the following spectral gap statement:

Proposition 3. Assume $u^{(0)}$ is generic satisfying the genericity conditions (i, ii). Let $\|a\|_\infty = \delta$. Then the operator F'_N has a spectral gap of size $\mathcal{O}(\delta^p/N)$ for some $q > b + 1$:

$$\|[F'_N]^{-1}\| \leq N/(\epsilon\delta^p), \quad q > b + 1, \quad \epsilon > 0, \quad (3.2)$$

on a set of $a \in (0, \delta]^b$ of measure at least $(1 - C\epsilon^c)\delta^b$, if $\delta N \ll 1$.

Remark. It will be clear from the proof that the same gap holds when also restricting the space frequency $j \in \mathbb{Z}^d$. So we do not state it separately.

Proof. Let P be the projection onto the bi-characteristics

$$\mathcal{C} = \{(n, j) \in \mathbb{Z}^{b+d} \mid \pm n \cdot \omega^{(0)} + \sqrt{j^2 + 1} = 0\}, \quad (3.3)$$

as defined in (2.6) and P^c onto the complement. Since $\omega^{(0)} = \{\sqrt{j_k^2 + 1}\}_{k=1}^b$ and $\sqrt{j^2 + 1}$ correspond to $b + 1$ algebraic numbers, if

$$\pm n \cdot \omega^{(0)} + \sqrt{j^2 + 1} \neq 0,$$

then

$$|\pm n \cdot \omega^{(0)} + \sqrt{j^2 + 1}| \geq c' \|n\|_1^{-q}, \quad (3.4)$$

for some $c' > 0$ and $q > b + 1$ [Schm], cf. also [R] for the scalar case. The bound in (3.4) implies that

$$\|[P^c F'_N P^c]^{-1}\| \leq C' N^q \quad (3.5)$$

for some $C' > 0$ and small δ .

From Schur's complement reduction, $\lambda \in \sigma(F'_N)$ if and only if $0 \in \sigma(\mathcal{H})$, where

$$\mathcal{H} = P F'_N P - \lambda + P F'_N P^c (P^c F'_N P^c - \lambda)^{-1} P^c F'_N P. \quad (3.6)$$

Moreover (3.5) implies that (3.6) is analytic in λ in the interval

$$(-1/(2C'N^q), 1/(2C'N^q))$$

and in the same interval

$$\|P F'_N P^c (P^c F'_N P^c - \lambda)^{-1} P^c F'_N P\| \leq \mathcal{O}(\delta^{2p} N^q). \quad (3.7)$$

Together with (2.3) and Proposition 1 this proves (3.2). \square

Proof of Theorem 1. To solve the nonlinear matrix equations (2.2), we use the Lyapunov-Schmidt decomposition as in [B2, 4, BW, CW, W1]. Let the resonant set $\mathcal{S} \subset \mathbb{Z}^{b+d}$

be as defined in (2.13). The equations restricted to $\mathbb{Z}^{b+d} \setminus \mathcal{S}$ are the P -equations and the rest are the Q -equations, which are in $2b$ dimensions.

Equations (2.2) are solved iteratively. The solution u is held fixed on \mathcal{S} :

$$u(\mp e_k, \pm j_k) = a_k, \quad k = 1, \dots, b.$$

We first solve the P -equations using a Newton scheme and at each step substitute the solution into the Q -equations to solve for the modulated b frequencies $\omega = \{\omega_k\}_{k=1}^b$.

We use a finite dimensional approximation to solve the P -equations. So the Newton scheme takes the form:

$$\Delta u^{(m+1)} = -F'_m(u^{(m)})^{-1} F_m(u^{(m)}), \quad (3.8)$$

where $u^{(m)}$ is the m^{th} approximation, $\Delta u^{(m+1)} = u^{(m+1)} - u^{(m)}$, $m = 0, 1, \dots$ and F_m and F'_m are appropriate finite dimensional approximations using as the initial approximation the finite subset $\Lambda = [-(\log \delta)^s, (\log \delta)^s]^{b+d}$ for some $s > 1$. The Q -equations give

$$\omega_k^{(m+1)} = \sqrt{j_k^2 + 1} + [F(u^{(m)})(-e_k, j_k)]/a_k, \quad k = 1, \dots, b. \quad (3.9)$$

Using Proposition 3, (3.8) gives after the first iteration $|\Delta u^{(1)}| = o(\delta^{3/2})$. As in [W1, Prop 2.2], (3.9) then gives the transversality condition

$$|\det(\frac{\partial \omega^{(1)}}{\partial a})| \asymp \delta^{pb}, \quad (3.10)$$

necessary for amplitude-frequency modulation. (3.10) enables us to apply the scheme of Bourgain for the b -parameter dependent nonlinear wave equations in [B4, Chap 20] and prove the theorem as in [W1]. \square

Proof of Theorem 2. Using Proposition 3, this follows the analysis scheme and the details in [W2]. \square

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